

Current Distribution in a Three-Dimensional IC Analyzed by a Perturbation Method

Part 1: A Simple Steady State Theory

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October 1, 1987



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Space Administration

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Pasadena, California

The research described in this publication was carried out by the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration.

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FOREWORD

The analysis in this publication was intended to be an analytical tool to assist a study of latchups in support of the Magellan space project. The study is not yet completed but it is expected that the analytical methods that have been developed so far may have other useful applications in the analysis of integrated circuits and therefore they are published here as a separate report. The analysis applies to the steady state current distribution in a three dimensional integrated circuit. A device physics approach--based on a perturbation method--rather than an equivalent lumped circuit approach is used. The perturbation method allows the various currents to be expressed in terms of "elementary solutions" which are solutions to very simple boundary value problems. Work is continuing and it is hoped that there will be a Part 2 report which applies the methods to the steady state characteristics of latched states. Examples which illustrate the use of the analytical methods are not given in this Part 1 report since this is expected to be discussed in detail in Part 2. Comparisons between theoretical predictions and measurements are also deferred to Part 2. The most obvious limitation of the present version of the analysis is that all depletion region boundary surfaces are treated as equipotential surfaces and this is why the report is subtitled, "A Simple Steady State Theory." This may be an adequate approximation in some applications but it is an obvious weakness in the theory when applied to latched states. This issue is expected to be confronted in Part 2.

ABSTRACT

The analysis applies to the steady state current distribution in a three dimensional integrated circuit. A device physics approach--based on a perturbation method--rather than an equivalent lumped circuit approach is used. The perturbation method allows the various currents to be expressed in terms of "elementary solutions" which are solutions to very simple boundary value problems.

The most obvious limitation of the present version of the analysis is that all depletion region boundary surfaces are treated as equipotential surfaces and this is why the report is subtitled, "A Simple Steady State Theory." This may be an adequate approximation in some applications but it is an obvious weakness in the theory when applied to latched states.

Examples that illustrate the use of these analytical methods are not given in this report because they will be presented in detail in a future report.

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SECTION 1

INTRODUCTION

The analysis in this report was intended to be an analytical tool to assist a study of latchups in support of the Magellan space project. An investigation of the application of this method of analysis to the steady state characteristics of latched states has not yet been completed. A time dependent version of this analytical method has also not yet been completed but work is continuing in these areas. So far, the analytical method, applicable to the steady state current distribution in a three dimensional integrated circuit, is reasonably well developed. Some work remains, as will be pointed out below, before it will be effective in the study of the steady state characteristics of latched states but it is expected that this analytical method will have many useful applications in its present stage of development. Therefore, it has been published here as a separate report. It is anticipated that there will be a Part 2 report which discusses the application of this theory to the steady state characteristics of latched states. Examples which demonstrate the use of the analytical method are not included in Part 1 since this will be treated in detail in Part 2. Comparisons between theoretical predictions and measurement are also deferred to Part 2. The most obvious limitation of the present version of the analysis is that all depletion region boundary surfaces are treated as equipotential surfaces and this is why this report is titled "...Part 1: A Simple Steady State Theory." This approximation is probably adequate in many applications but it is a definite weakness in the theory when applied to latched states. If there is not a more elegant way to deal with the boundary values at a depletion region, we could at least partition it into sections with each section treated as an equipotential surface. This would be an improvement over treating the entire depletion region surface as an equipotential. However, such a partitioning would require that we correctly represent the tangential currents that flow inside the depletion region between adjacent sections and it is not obvious at this time how to do this. This issue will be confronted in Part 2. Finally, it is hoped that there will eventually be a Part 3 report which discusses transient conditions.

The customary method of analyzing the behavior of a real (three dimensional) integrated circuit is to simulate the device with an equivalent lumped element circuit. A problem with using a lumped circuit to simulate a distributed system is that the interactions between various regions in a device are not the same as they would be if the various regions were separated and connected by wires. It is sometimes difficult to determine representative values for the circuit elements used in an equivalent circuit. This is especially true for the gains of parasitic transistors when the device is in a latched state [1]. These problems can be avoided by treating the device as a distributed system, which is what it is. A difficulty with this approach is that the governing equations are nonlinear and difficult to solve unless some approximations can be made. An approximation proposed in this report is to treat the electric field as a perturbation, that can be treated by first order perturbation theory, in the quasineutral regions of the device. This produces a system of linear differential equations in the quasineutral regions and a superposition principle will be used to construct solutions out of "elementary" solutions. The elementary solutions satisfy very simple boundary value problems. Boundary conditions, which represent the effects of various kinds of junctions (e.g., depletion regions, high-low junctions, etc.), will have to be satisfied and this will result in a system of nonlinear equations. However, these equations will be algebraic rather than differential equations. The elementary solutions will be used to construct various constants (the constants will be surface or volume integrals of these solutions). These constants will be analogous to device parameters (e.g., resistances) used in an equivalent circuit simulation and they can be evaluated either experimentally or by analytically or numerically solving very simple boundary value problems. With the constants regarded as known quantities, the algebraic equations which represent the boundary conditions can be solved.

SECTION 2

APPROXIMATION FOR THE ELECTRIC FIELD

Consider a quasineutral region which is denoted by R and assumed to be homogeneous. The well-known equations for the electron and hole currents are

$$\vec{J}_p = eD_p \left[-\vec{\nabla}_p + p \frac{\vec{E}}{V_T} \right] \quad (1a)$$

$$\vec{J}_n = eD_n \left[\vec{\nabla}_n + n \frac{\vec{E}}{V_T} \right] \quad (1b)$$

where $V_T = KT/e$ is the thermal voltage and the other symbols have the obvious interpretations. Since R is a quasineutral region, we assume that the "applied" electric fields are much stronger than the fields associated with space charge separations in R and we make the approximation that the field \vec{E} in (1) is the applied field. This applied field is associated with surface charges induced by an external power supply rather than charges on the interior of the homogeneous region R. So the first approximation that is made is the assumption that the electric field in (1) has a zero divergence in R. This approximation (which obviously does not apply inside transition regions) is nothing new. Current induced voltage drops are typically solved by assuming ohmic drops and solving Laplace's equation [2]. This is equivalent to the assumption of zero charge densities in the quasineutral regions. However, "applied" field has not yet been precisely defined. The only quantitative statement made so far is that it has a zero divergence in R. It is not completely specified until boundary values are specified for the potential that the field is derived from. Approximations that will be used for the boundary values will be discussed in the next section but, ideally, the potential that the electric field is derived from is to be the same as the "applied potential" which is defined more rigorously in discussions of quasi-Fermi levels. Working with this kind of potential makes it unnecessary

to subtract out the built-in equilibrium potentials associated with each transition region in order to compute terminal voltages.

An additional approximation will be the use of first order perturbation theory, with the electric field regarded as the perturbation. There are intuitive reasons why we might expect first order perturbation theory to work, but the reasons applicable to majority carriers are different than the reasons applicable to minority carriers. In the case of minority carriers, the drift current is typically small compared to the diffusion current since most of the drift current is carried by the majority carriers. In many situations, the drift term in the minority current can be completely neglected [3]. In situations where it can't be completely neglected, it is reasonable to expect that it can be treated as a small perturbation. In the case of majority carriers, the drift term can't be treated as small since it is often the dominant term. However, the success of first order perturbation theory does not necessarily require that the perturbation be small, it does require that the higher order terms in the expansion of the unknown function be small compared to the linear terms. The customary use of simple resistances to calculate current induced voltage drops suggests that the majority current is approximately a linear function of the applied electric field (this is not the same as saying that currents are linear functions of terminal voltages. Nonlinearities result from boundary values at various junctions) and this suggests that first order expansions may be adequate.

Although the electric field in (1) will be approximated as being the applied field, the field is still an unknown (if it were known, perturbation methods would be unnecessary because the equations would already be linear) because some of the boundary values that control it are unknown. In fact, these boundary values will be the parameters used in the perturbation expansion. Therefore, boundary values should be discussed before the details of the perturbation method are discussed.

SECTION 3

BOUNDARY CONDITIONS

The boundary surface of the region R is assumed to be composed of discrete sections where any given section is either an insulated surface (oxide layer or semiconductor to free space junction), depletion region surface, metal to semiconductor contact, or a high-low junction.

For each boundary surface that separates two interacting regions, the boundary conditions will be characterized in two different ways. The first characterization states the general class (e.g., Dirichlet, Neumann or mixed) that the boundary conditions fall in. A typical example of such a characterization is the statement that the potential and carrier densities are assumed constant on a given surface. The values of these constants, i.e., the boundary values, are often unknowns and must be adjusted to satisfy the second characterization (discussed below). The first characterization will be used when working with the differential equations to express solutions in terms of the unknown boundary values. Later, after the equations have been reduced to algebraic equations, the second characterization is to be used to complete the solution. The second characterization relates boundary values on opposite sides of a given surface. If all potentials were known quantities and we are interested in solving for majority and minority currents and/or densities, four equations are needed for each boundary surface. In their most simple form, the equations would be specifications of both kinds of carrier densities on both sides of the surface (in this context, "surface" refers to a junction even though it has a finite thickness so carrier densities can be different on different sides of the surface). More generally, the equations could include a relationship between carrier densities on opposite sides of the surface and a relationship between current densities on opposite sides of the surface. Since the potentials are also unknowns that must be solved, the boundary conditions must also contain enough information to specify the potential on both sides of the boundary surface. Altogether we will need six equations for each boundary surface that separates interacting regions. The surfaces of this type that are discussed in this report are high-low junctions and

depletion regions. Surfaces that are not of this type are insulated surfaces and metal to semiconductor contacts. For these surfaces it is only necessary to solve the equations on one side of the surface. Therefore, only three boundary conditions (applicable to the relevant side of the surface) must be given. This information is built into the first characterization of the boundary conditions so only the first characterization is needed for such surfaces.

A. INSULATED SURFACES

To find the boundary conditions for the potential, we use the same kind of reasoning that would be used if Ohm's law could be applied to the region R. There will be surface charges on the boundary surfaces of R associated with the applied voltages. If we look within the surface charge layer we would see a normal component of the gradient of the charge density and we would see drift currents and diffusion currents opposing each other and Ohm's law would not apply for any kind of material. Furthermore, the potential does not satisfy Laplace's equation here. But if we look just below this surface charge layer (in this context, "below" means locations on the interior of R that are very close to the surface charge layer) and assume Ohm's law to apply, we would conclude that the electric field has a zero normal component at these locations. Therefore, the boundary condition that will be used for the applied potential is

$$\vec{\nabla}\phi(\vec{X}) \cdot \hat{K} = 0 \quad (2)$$

where \hat{K} is the unit vector perpendicular to the surface. Note that the boundary condition (2) is expected to provide an approximation for the potential on the interior of R but this region excludes the surface charge layers. The currents calculated from this potential exclude the tangential currents flowing through the surface charge layers. In particular, the current under the insulated gate of a MOSFET is excluded. Since this current can be important, it must be included as a separate term.

Boundary conditions for the carrier densities will make use of the familiar recombination velocity which gives

$$\frac{\vec{J}_p}{e} \cdot \hat{K} = S(p - p_o) = -\frac{\vec{J}_n}{e} \cdot \hat{K} \quad \text{for n type}$$

$$\frac{\vec{J}_p}{e} \cdot \hat{K} = S(n - n_o) = -\frac{\vec{J}_n}{e} \cdot \hat{K} \quad \text{for p type}$$

where n_o and p_o are the equilibrium carrier densities in R and \hat{K} is the outer normal unit vector and S is the recombination velocity. The first equations are used if R is composed of n type material and the second are used if R is p type. Since the drift currents are assumed to be tangential to the boundary surface, the above equations become

$$-D_p \vec{\nabla}_p \cdot \hat{K} = S(p - p_o) = -D_n \vec{\nabla}_n \cdot \hat{K} \quad \text{for n type} \quad (3a)$$

$$-D_p \vec{\nabla}_p \cdot \hat{K} = S(n - n_o) = -D_n \vec{\nabla}_n \cdot \hat{K} \quad \text{for p type.} \quad (3b)$$

B. DEPLETION REGIONS

The two boundary surfaces of a depletion region are each assumed to be a constant potential, constant n, constant p surface. Using these approximations, boundary conditions are of the form

$$\phi(\vec{X}) = \phi = \text{constant} \quad (4a)$$

$$n(\vec{X}) = n = \text{constant} \quad (4b)$$

$$p(\vec{X}) = p = \text{constant.} \quad (4c)$$

The boundary values are not independent and at some stage in the analysis they must be adjusted to satisfy certain conditions. If the junction is abrupt and

if recombination in the space charge region can be neglected, it is shown in Appendix A that the conditions are

$$p_p = \frac{\frac{n_{on}}{n_{op}} \beta^{-1} N_{Ap} + N_{Dn}}{\frac{n_{on}}{n_{op}} \beta^{-1} - \frac{p_{on}}{p_{op}} \beta} \quad (5a)$$

$$p_n = \frac{\frac{n_{op}}{n_{on}} \beta N_{Dn} + N_{Ap}}{\frac{p_{op}}{p_{on}} \beta^{-1} - \frac{n_{op}}{n_{on}} \beta} \quad (5b)$$

$$n_p = \frac{\frac{p_{on}}{p_{op}} \beta N_{Ap} + N_{Dn}}{\frac{n_{on}}{n_{op}} \beta^{-1} - \frac{p_{on}}{p_{op}} \beta} \quad (5c)$$

$$n_n = \frac{\frac{p_{op}}{p_{on}} \beta^{-1} N_{Dn} + N_{Ap}}{\frac{p_{op}}{p_{on}} \beta^{-1} - \frac{n_{op}}{n_{on}} \beta} \quad (5d)$$

where

$$\beta = \exp [(\phi_p - \phi_n)/V_T]. \quad (6)$$

N_{Dn} and N_{Ap} are the donor density on the n side and the acceptor density on the p side. The hole and electron carrier densities on the p and n side are denoted by p_p , p_n , n_p and n_n while the equilibrium values of these densities are p_{op} , p_{on} , n_{op} and n_{on} . The potentials on the two sides are ϕ_p and ϕ_n . The equilibrium densities can be solved from

$$p_{on} = \frac{n_i^2}{N_{Dn}} \exp [\Delta V_n/V_T] \quad (7a)$$

$$n_{op} = \frac{n_i^2}{N_{Ap}} \exp [\Delta V_p / V_T] \quad (7b)$$

$$p_{op} = N_{Ap} + \frac{n_i^2}{N_{Ap}} \exp [\Delta V_p / V_T] \quad (7c)$$

$$n_{on} = N_{Dn} + \frac{n_i^2}{N_{Dn}} \exp [\Delta V_n / V_T] \quad (7d)$$

where n_i is the intrinsic carrier density (the density in a pure material) and ΔV_n and ΔV_p are the values of ΔV_{go} in the n material and p material respectively with ΔV_{go} an empirically determined function of the doping concentration. Empirical information on ΔV_{go} has been provided by Slotboom and De Graaff [4].

To the equations we must add

$$\vec{J}_p \cdot \hat{K} \Big|_{n \text{ side}} = \vec{J}_p \cdot \hat{K} \Big|_{p \text{ side}} \quad (8a)$$

$$\vec{J}_n \cdot \hat{K} \Big|_{n \text{ side}} = \vec{J}_n \cdot \hat{K} \Big|_{p \text{ side}} \quad (8b)$$

C. HIGH-LOW JUNCTIONS

The two boundary surfaces of a high-low junction are each assumed to be a constant potential, constant n, constant p surface so the boundary conditions are of the form (4).

As with the depletion region, the boundary values must satisfy certain conditions. If the junction is abrupt and if recombination in the space charge region can be neglected, it is shown in Appendix A that these conditions for an n-n⁺ junction are

$$p_{\ell} = \frac{\frac{n_{oh}}{n_{ol}} \beta^{-1} N_{D\ell} - N_{Dh}}{\frac{p_{oh}}{p_{ol}} \beta - \frac{n_{oh}}{n_{ol}} \beta^{-1}} \quad (9a)$$

$$p_h = \frac{\frac{n_{ol}}{n_{oh}} \beta N_{Dh} - N_{D\ell}}{\frac{p_{ol}}{p_{oh}} \beta^{-1} - \frac{n_{ol}}{n_{oh}} \beta} \quad (9b)$$

$$n_{\ell} = \frac{\frac{-p_{oh}}{p_{ol}} \beta N_{D\ell} + N_{Dh}}{\frac{n_{oh}}{n_{ol}} \beta^{-1} - \frac{p_{oh}}{p_{ol}} \beta} \quad (9c)$$

$$n_h = \frac{\frac{-p_{ol}}{p_{oh}} \beta^{-1} N_{Dh} + N_{D\ell}}{\frac{n_{ol}}{n_{oh}} \beta - \frac{p_{ol}}{p_{oh}} \beta^{-1}} \quad (9d)$$

$$\beta = \exp [(\phi_{\ell} - \phi_h)/V_T] \quad (10)$$

$$p_{ol} = \frac{n_i^2}{N_{D\ell}} \quad (11a)$$

$$p_{oh} = \frac{n_i^2}{N_{Dh}} \exp [\Delta V_{go}/V_T] \quad (11b)$$

$$n_{ol} = N_{D\ell} + \frac{n_i^2}{N_{D\ell}} \quad (11c)$$

$$n_{oh} = N_{Dh} + \frac{n_i^2}{N_{Dh}} \exp [\Delta V_{go}/V_T] . \quad (11d)$$

The symbols are the same as with the p_n junction except that subscripts l and h , which refer to the lightly doped and heavily doped regions, replace p and n . ΔV_{go} refers only to the heavily doped side. For a $p-p^+$ junction, the equations are

$$p_l = \frac{-\frac{n_{oh}}{n_{ol}} \beta^{-1} N_{Al} + N_{Ah}}{\frac{p_{oh}}{p_{ol}} \beta - \frac{n_{oh}}{n_{ol}} \beta^{-1}} \quad (12a)$$

$$p_h = \frac{-\frac{n_{ol}}{n_{oh}} \beta N_{Ah} + N_{Al}}{\frac{p_{ol}}{p_{oh}} \beta^{-1} - \frac{n_{ol}}{n_{oh}} \beta} \quad (12b)$$

$$n_l = \frac{\frac{p_{oh}}{p_{ol}} \beta N_{Al} - N_{Ah}}{\frac{n_{oh}}{n_{ol}} \beta^{-1} - \frac{p_{oh}}{p_{ol}} \beta} \quad (12c)$$

$$n_h = \frac{\frac{p_{ol}}{p_{oh}} \beta^{-1} N_{Ah} - N_{Al}}{\frac{n_{ol}}{n_{oh}} \beta - \frac{p_{ol}}{p_{oh}} \beta^{-1}} \quad (12d)$$

$$\beta = \exp [(\phi_l - \phi_h)/V_T] \quad (13)$$

$$n_{ol} = \frac{n_i^2}{N_{Al}} \quad (14a)$$

$$n_{oh} = \frac{n_i^2}{N_{Ah}} \exp [\Delta V_{go}/V_T] \quad (14b)$$

$$p_{ol} = N_{Al} + \frac{n_i^2}{N_{Al}} \quad (14c)$$

$$p_{oh} = N_{Ah} + \frac{n_i^2}{N_{Ah}} \exp [\Delta V_{go}/V_T] . \quad (14d)$$

In both cases we have

$$\vec{J}_p \cdot \hat{K} \Big|_{\text{high side}} = \vec{J}_p \cdot \hat{K} \Big|_{\text{low side}} \quad (15a)$$

$$\vec{J}_n \cdot \hat{K} \Big|_{\text{high side}} = \vec{J}_n \cdot \hat{K} \Big|_{\text{low side}} . \quad (15b)$$

D. METAL TO SEMICONDUCTOR CONTACT (OHMIC)

Metal to semiconductor contacts will be assumed to be equipotential surfaces with an infinite recombination velocity. Boundary conditions are of the form

$$\phi(\vec{X}) = \phi' = \text{constant} \quad (16a)$$

$$n(\vec{X}) = n_o \quad (16b)$$

$$p(\vec{X}) = p_o . \quad (16c)$$

If the contact resistance can be neglected, ϕ' in (16a) is the potential of the contact lead. If this resistance must be included, it can be simulated by placing an imaginary resistor between the contact and the contact lead. In this case, ϕ' would be the potential on the contact side of the resistor.

SECTION 4

CALCULATION OF CARRIER DENSITIES IN TERMS OF BOUNDARY VALUES

This section will relate carrier densities to the boundary values by assuming boundary conditions to be of the form (2), (3) or (4). Some boundary values will usually be unknown and must be adjusted to satisfy equations like (5), (8), (9) and (15) but these adjustments are to be made after algebraic equations that relate currents to boundary values are obtained.

To be definite, suppose the region R is an n type material. Note that all boundary surfaces except the insulated surfaces specify Dirichlet boundary conditions for ϕ , p and n. Imagine the boundary surface of R to be the union of surface sections with the i^{th} section denoted by A_i . Let INS and OTH be index sets such that:

If $i \in \text{INS}$ then A_i is an insulated surface with boundary conditions

$$\vec{\nabla} \phi \cdot \hat{K} = 0, \quad -D_p \vec{\nabla} p \cdot \hat{K} = S_i(p - p_0), \quad -D_n \vec{\nabla} n \cdot \hat{K} = S_i(p - p_0).$$

If $i \in \text{OTH}$ then A_i is any surface other than an insulated surface and has boundary conditions $\phi = V_i$, $n = n_i$, $p = p_i$.

In this section, n_i and p_i refer to boundary values on the i^{th} surface rather than intrinsic densities.

Define

$$N = n - n_0, \quad P = p - p_0.$$

Using (1) and the equations [3]

$$\vec{\nabla} \cdot \vec{J}_p = -\frac{e(p - p_0)}{\tau}, \quad \vec{\nabla} \cdot \vec{J}_n = \frac{e(p - p_0)}{\tau}, \quad \vec{\nabla} \cdot \vec{E} = 0$$

gives

$$\nabla^2 P - \frac{1}{V_T} \vec{E} \cdot \vec{\nabla} P = \frac{P}{L_p^2} \quad (17a)$$

$$\nabla^2 N + \frac{1}{V_T} \vec{E} \cdot \vec{\nabla} N = \frac{P}{L_n^2} \quad (17b)$$

where $L_p = (D_p \tau)^{1/2}$, $L_n = (D_n \tau)^{1/2}$ are the diffusion lengths for holes and electrons and τ is the recombination time. The boundary conditions can be expressed as

If $i \in \text{INS}$ and \vec{X} on A_i then

$$\vec{\nabla} \phi \cdot \hat{K} = 0 \quad (18a)$$

$$\vec{\nabla} P \cdot \hat{K} = \frac{-S_i \tau}{L_p^2} P \quad (18b)$$

$$\vec{\nabla} N \cdot \hat{K} = \frac{-S_i \tau}{L_n^2} P. \quad (18c)$$

If $i \in \text{OTH}$ and \vec{X} on A_i then

$$\phi = V_i \quad (19a)$$

$$P = P_i \quad (19b)$$

$$N = N_i. \quad (19c)$$

For each $j \in \text{OTH}$, define a function $\phi_j(\vec{X})$ by the conditions

$$\nabla^2 \phi_j = 0 \text{ in } R \quad (20a)$$

$$\text{If } i \in \text{INS} \text{ and } \vec{X} \text{ on } A_i \text{ then } \vec{\nabla} \phi_j \cdot \hat{K} = 0 \quad (20b)$$

$$\text{If } i \in OTH \text{ and } \vec{X} \text{ on } A_i \text{ then } \phi_j = \delta_{ij} \quad (20c)$$

where δ_{ij} is the Kronecker delta. Note that the sum

$$\sum_{j \in OTH} v_j \phi_j(\vec{X})$$

satisfies Laplace's equation in R and the boundary conditions imposed on the potential ϕ . It is well known that solutions to such problems are unique so we have

$$\phi(\vec{X}) = \sum_{j \in OTH} v_j \phi_j(\vec{X}) . \quad (21)$$

The ϕ_i 's are not all independent. The sum

$$\sum_{j \in OTH} \phi_j(\vec{X})$$

is one on every noninsulated surface and has a zero normal gradient on the insulated surfaces. The constant value 1 satisfies Laplace's equation and also has these properties which implies

$$\sum_{j \in OTH} \phi_j(\vec{X}) = 1 . \quad (22)$$

Constraint (22) is consistent with the well known fact that any potential can be chosen as a reference and set equal to zero, which eliminates one of the functions that must be included on the right side of (21). If none of the potentials are set equal to zero, (22) can still be used to reduce the number of boundary value problems that must be solved in order to obtain all of the ϕ_i 's.

Using (21), the electric field is given by

$$\vec{E} = -\vec{\nabla} \phi = - \sum_{j \in \text{OTH}} V_j \vec{\nabla} \phi_j \quad (23)$$

so (17) becomes

$$\nabla^2 P + \frac{1}{V_T} \vec{\nabla} P \cdot \sum_{j \in \text{OTH}} V_j \vec{\nabla} \phi_j = \frac{P}{L_p^2} \quad (24a)$$

$$\nabla^2 N - \frac{1}{V_T} \vec{\nabla} N \cdot \sum_{j \in \text{OTH}} V_j \vec{\nabla} \phi_j = \frac{P}{L_n^2} \quad (24b)$$

Equations (24) and the boundary conditions (18) and (19) define P and N as a function of the V_i 's, i.e.,

$$P = P(\vec{X}, V_i), N = N(\vec{X}, V_i) \quad .$$

It is assumed that the functions can be expressed in a series of the form

$$P(\vec{X}, V_i) = \rho^{(0)}(\vec{X}) + \sum_{j \in \text{OTH}} \rho_j(\vec{X}) V_j + O_1(2) \quad (25a)$$

$$N(\vec{X}, V_i) = \eta^{(0)}(\vec{X}) + \sum_{j \in \text{OTH}} \eta_j(\vec{X}) V_j + O_2(2) \quad (25b)$$

where $O_1(2)$ and $O_2(2)$ represent terms of second or higher order in the V_i 's. Parentheses are used to emphasize that the superscripts are indices rather than exponents. Substituting (25) into (24) and equating coefficients to like powers of the V_i 's yield

$$\nabla^2 \rho^{(0)} = \frac{\rho^{(0)}}{L_p^2} \quad (26)$$

$$\nabla^2 \rho_j + \frac{1}{v_T} \vec{\nabla} \rho^{(o)} \cdot \vec{\nabla} \phi_j = \frac{\rho_j}{L_p^2} \quad (27)$$

$$\nabla^2 \eta^{(o)} = \frac{\rho^{(o)}}{L_n^2} \quad (28)$$

$$\nabla^2 \eta_j - \frac{1}{v_T} \vec{\nabla} \eta^{(o)} \cdot \vec{\nabla} \phi_j = \frac{\rho_j}{L_n^2} . \quad (29)$$

By using the identity

$$\nabla^2 (fg) = 2\vec{\nabla} f \cdot \vec{\nabla} g + f\nabla^2 g + g\nabla^2 f$$

together with $\nabla^2 \phi = 0$ and (26), we can express (27) as

$$\nabla^2 [\rho_j + \frac{1}{2v_T} \rho^{(o)} \phi_j] = \frac{1}{L_p^2} [\rho_j + \frac{1}{2v_T} \rho^{(o)} \phi_j] . \quad (30)$$

Also, (26) and (28) can be combined to give

$$\nabla^2 [\eta^{(o)} - \frac{L_p^2}{L_n^2} \rho^{(o)}] = 0 . \quad (31)$$

To obtain boundary conditions satisfied by the η 's and ρ 's, first consider an insulated surface A_i . From (18b) and (25a) we have, for \vec{X} on A_i ,

$$\begin{aligned} & [\vec{\nabla} \rho^{(o)} \cdot \hat{K} + \frac{S_i \tau}{L_p^2} \rho^{(o)}] + \sum_j v_j [\vec{\nabla} \rho_j \cdot \hat{K} + \frac{S_i \tau}{L_p^2} \rho_j] \\ & + [\vec{\nabla} \eta_1(2) \cdot \hat{K} + \frac{S_i \tau}{L_p^2} \eta_1(2)] = 0 . \end{aligned}$$

This is to be satisfied identically in the V 's which gives

$$\vec{V}_\rho^{(o)} \cdot \hat{K} = \frac{-S_i \tau}{L_p^2} \rho^{(o)}, \quad \vec{V}_{\rho_j} \cdot \hat{K} = \frac{-S_i \tau}{L_p^2} \rho_j$$

with analogous equations for the η 's. Note that (25a) and the requirement that (19b) be satisfied identically in the V_i 's give

$$\rho^{(o)} = P_i, \quad \rho_j = 0$$

on the noninsulated boundary surfaces. The boundary conditions are summarized below.

For each $j \in OTH$ we have:

If $i \in INS$ and \vec{X} on \vec{A}_i then

$$\vec{V}_\rho^{(o)} \cdot \hat{K} = \frac{-S_i \tau}{L_p^2} \rho^{(o)} \quad (32a)$$

$$\vec{V}_{\rho_j} \cdot \hat{K} = \frac{-S_i \tau}{L_p^2} \rho_j \quad (32b)$$

$$\vec{V}_\eta^{(o)} \cdot \hat{K} = \frac{-S_i \tau}{L_n^2} \rho^{(o)} \quad (32c)$$

$$\vec{V}_{\eta_j} \cdot \hat{K} = \frac{-S_i \tau}{L_n^2} \rho_j. \quad (32d)$$

If $i \in OTH$ and \vec{X} on A_i then

$$\rho^{(o)} = P_i \quad (33a)$$

$$\rho_j = 0 \quad (33b)$$

$$\eta^{(o)} = N_i \quad (33c)$$

$$\eta_j = 0 . \quad (33d)$$

For each $j \in 0TH$, define the function ψ_j by the conditions:

$$\nabla^2 \psi_j = \frac{\psi_i}{L_p^2} \text{ in } R . \quad (34)$$

If $i \in INS$ and \vec{X} on A_i then

$$\vec{\nabla} \psi_j \cdot \hat{K} = \frac{-S_i \tau}{L_p^2} \psi_j . \quad (35)$$

If $i \in 0TH$ and \vec{X} on A_i then

$$\psi_j = \delta_{ij} . \quad (36)$$

It is evident that the sum

$$\sum_{j \in 0TH} P_j \psi_j$$

satisfies the same field equation and boundary conditions as $\rho^{(o)}$. It is shown in Appendix B that solutions to such problems are unique which implies

$$\rho^{(o)} = \sum_{j \in 0TH} P_j \psi_j . \quad (37)$$

From (30) it is seen that

$$\rho_j + \frac{1}{2V_T} \rho^{(o)} \phi_j$$

satisfies the same field equation as ψ_j . Note

$$\vec{\nabla} \left[\rho_j + \frac{1}{2V_T} \rho^{(o)} \phi_j \right] \cdot \hat{K} = \vec{\nabla} \rho_j \cdot \hat{K} + \frac{1}{2V_T} \rho^{(o)} \vec{\nabla} \phi_j \cdot \hat{K} + \frac{1}{2V_T} \phi_j \vec{\nabla} \rho^{(o)} \cdot \hat{K}.$$

Using (32b), (20b) and (32a) shows that on the insulated surfaces,

$$\vec{\nabla} \left[\rho_j + \frac{1}{2V_T} \rho^{(o)} \phi_j \right] \cdot \hat{K} = \frac{-S_i \tau}{L_p^2} \rho_j - \frac{\phi_j}{2V_T} \frac{S_i \tau \rho^{(o)}}{L_p^2} = \frac{-S_i \tau}{L_p^2} \left[\rho_j + \frac{1}{2V_T} \rho^{(o)} \phi_j \right]$$

which is the same boundary condition satisfied by ψ_j on the insulated surfaces. Also, on a noninsulated surface A_i , we have from (33a), (33b) and (20c) that

$$\rho_j + \frac{1}{2V_T} \rho^{(o)} \phi_j = \frac{P_i}{2V_T} \delta_{ij} = \frac{P_j}{2V_T} \delta_{ij}$$

which are the same boundary conditions satisfied by

$$\frac{P_j}{2V_T} \psi_j.$$

We therefore have

$$\rho_j + \frac{1}{2V_T} \rho^{(o)} \phi_j = \frac{P_j}{2V_T} \psi_j$$

or, using (37),

$$\rho_j = \frac{1}{2V_T} \left[P_j \psi_j - \phi_j \sum_{K \in OTH} P_K \psi_K \right]. \quad (38)$$

Now consider the expression

$$\eta^{(o)} - \frac{L_p^2}{L_n^2} \rho^{(o)}.$$

It is seen from (31) that this expression satisfies Laplace's equation inside R and from (32a) and (32c) it is seen that on the insulated surfaces,

$$\vec{\nabla} \left[\eta^{(o)} - \frac{L_p^2}{L_n^2} \rho^{(o)} \right] \cdot \hat{K} = 0.$$

From (33a) and (33c) it is seen that on a noninsulated surface A_i we have

$$\eta^{(o)} - \frac{L_p^2}{L_n^2} \rho^{(o)} = N_i - \frac{L_p^2}{L_n^2} P_i.$$

The expression

$$\sum_{j \in OTH} \left(N_j - \frac{L_p^2}{L_n^2} P_j \right) \phi_j$$

also has these properties so we conclude

$$\eta^{(o)} - \frac{L_p^2}{L_n^2} \rho^{(o)} = \sum_{j \in OTH} \left(N_j - \frac{L_p^2}{L_n^2} P_j \right) \phi_j$$

or, using (37),

$$\begin{aligned}\eta^{(o)} &= \frac{L_p^2}{L_n^2} \sum_{j \in \text{OTH}} P_j \psi_j + \sum_{j \in \text{OTH}} \left(N_j - \frac{L_p^2}{L_n^2} P_j \right) \phi_j \\ &= \sum_{j \in \text{OTH}} \left[N_j \phi_j + \frac{L_p^2}{L_n^2} P_j (\psi_j - \phi_j) \right].\end{aligned}\tag{39}$$

Equations (37), (38) and (39) solve for $\rho^{(o)}$, ρ_j and $\eta^{(o)}$ in terms of the elementary functions ψ_j and ϕ_j . If the second term on the left side of (29) contained a plus sign instead of a minus sign, η_j would be easy to solve using the same basic methods that were used to solve for the other quantities. As it is, η_j is not easy to solve. However, it is unlikely that we would be interested in the majority carrier density away from the boundary surfaces except for the purpose of calculating the gradient in order to obtain the diffusion current. Fortunately, as will be shown in the next section, it is not necessary to solve (29) to do this.

The important equations in this section and the analogous equations that apply to a p type material are listed in the summary table which follows the next section.

SECTION 5

EXPRESSING CURRENTS IN TERMS OF THE BOUNDARY VALUES

The previous section related carrier densities to the boundary values (some of which will usually be unknown) and the elementary functions ϕ_j, ψ_j . This section will do a similar thing for the currents through a given noninsulated surface. The minority carrier recombination current at an insulated surface is also easy to express in terms of the elementary functions (this also solves for the majority carrier recombination current since the net current is zero) but these currents have no immediate use in the analysis presented here. Therefore, we will confine our attention to the noninsulated boundary surfaces.

For notational brevity in the equations that will follow, some constants will now be defined. For each i and j in OTH, define Q_{ij} and R_{ij} by

$$Q_{ij} = \int_{A_i} \vec{\nabla} \psi_j \cdot d\vec{s} \quad (40)$$

$$R_{ij} = \int_{A_i} \vec{\nabla} \phi_j \cdot d\vec{s} \quad (41)$$

where the integrals are surface integrals on the A_i surface and the unit normal vector is in the direction of \hat{K} , i.e., it is directed outwards from the region R .

R_{ij} is easily recognized as being proportional to the current in an Ohm's law medium, through the surface A_i , divided by the potential on the surface A_j when all noninsulated surfaces except A_j are grounded. R_{ij} is therefore inversely proportional to an electrical resistance when the region R contains an ohmic material. Q_{ij} is proportional to the current in a pure diffusive medium (a medium that supports diffusion currents but not drift currents) through the surface A_i divided by the excess carrier density on the surface A_j with the excess carrier density zero on all other noninsulated surfaces. Therefore, Q_{ij} can be thought of as the reciprocal of a diffusion resistance.

The constants Q_{ij} and R_{ij} are not all independent, there are several constraints connecting them. For example, using (36) gives

$$Q_{ij} = \int_{A_i} \vec{\nabla} \psi_j \cdot d\vec{s} = \sum_{k \in OTH} \int_{A_k} \psi_i \vec{\nabla} \psi_j \cdot d\vec{s} = \oint \psi_i \vec{\nabla} \psi_j \cdot d\vec{s} \\ - \sum_{k \in INS} \int_{A_k} \psi_i \vec{\nabla} \psi_j \cdot d\vec{s} .$$

But (35) gives

$$\int_{A_k} \psi_i \vec{\nabla} \psi_j \cdot d\vec{s} = \frac{-S_k \tau}{L_p^2} \int_{A_k} \psi_i \psi_j ds$$

and Green's first identity gives

$$\oint \psi_i \vec{\nabla} \psi_j \cdot d\vec{s} = \int_R [\psi_i \nabla^2 \psi_j + \vec{\nabla} \psi_i \cdot \vec{\nabla} \psi_j] \cdot d\vec{s} .$$

Using (34) gives

$$\oint \psi_i \vec{\nabla} \psi_j \cdot d\vec{s} = \int_R \left[\frac{\psi_i \psi_j}{L_p^2} + \vec{\nabla} \psi_i \cdot \vec{\nabla} \psi_j \right] d^3X$$

so that

$$Q_{ij} = \int_R \left[\frac{\psi_i \psi_j}{L_p^2} + \vec{\nabla} \psi_i \cdot \vec{\nabla} \psi_j \right] d^3X + \sum_{k \in INS} \frac{S_k \tau}{L_p^2} \int_{A_k} \psi_i \psi_j ds .$$

The expression on the right of the above equation is symmetric in i, j which proves the reciprocity relation

$$Q_{ij} = Q_{ji} . \quad (42)$$

A simpler version of the above proof will prove the well known reciprocity relation for the electrostatic potential which, in this notation, is written as

$$R_{ij} = R_{ji} . \quad (43)$$

Another constraint can be obtained by taking the surface integral of the gradient of both sides of (22) on any noninsulated surface A_i . The result is

$$\sum_{j \in \text{OTH}} \int_{A_i} \vec{\nabla} \phi_j \cdot d\vec{s} = 0$$

or

$$\sum_{j \in \text{OTH}} R_{ij} = 0 \quad \text{for any } i \in \text{OTH} \quad (44)$$

Equation (44) could also have been deduced by using (20b) and recognizing that $\vec{\nabla} \phi_j$ has a zero surface integral on a closed surface.

The hole current directed outward through a noninsulated surface A_i is given by

$$I_{P_i} = \int_{A_i} \vec{J}_P \cdot d\vec{s} = e D_P \int_{A_i} \left[-\vec{\nabla} P + \frac{(P + P_o)}{V_T} \vec{E} \right] \cdot d\vec{s} .$$

Using (19b) gives

$$\frac{I_{P_i}}{e D_P} = - \int_{A_i} \vec{\nabla} P \cdot d\vec{s} + \frac{P_i + P_o}{V_T} \int_{A_i} \vec{E} \cdot d\vec{s} .$$

To first order in the V's we set

$$P = \rho^{(0)} + \sum_{j \in \text{OTH}} V_j \rho_j .$$

Using (23), (37) and (38) gives

$$\begin{aligned} \int_{A_i} \vec{\nabla} P \cdot d\vec{s} &= \sum_j P_j \int_{A_i} \vec{\nabla} \psi_j \cdot d\vec{s} \left[1 + \frac{V_j}{2V_T}\right] - \sum_j \sum_k \frac{V_j P_k}{2V_T} \int_{A_i} \psi_k \vec{\nabla} \phi_j \cdot d\vec{s} \\ &\quad - \sum_j \sum_k \frac{V_j P_k}{2V_T} \int_{A_i} \phi_j \vec{\nabla} \psi_k \cdot d\vec{s} \end{aligned}$$

and

$$\int_{A_i} \vec{E} \cdot d\vec{s} = - \sum_j V_j \int_{A_i} \vec{\nabla} \phi_j \cdot d\vec{s}.$$

Using (20c), (36), (40) and (41), the equations reduce to

$$\int_{A_i} \vec{\nabla} P \cdot d\vec{s} = \sum_j P_j \left[1 + \frac{V_j - V_i}{2V_T}\right] Q_{ij} - P_i \sum_j \frac{V_j}{2V_T} R_{ij} \quad (45)$$

$$\int_{A_i} \vec{E} \cdot d\vec{s} = - \sum_j V_j R_{ij} \quad (46)$$

so that

$$\frac{I_{P_i}}{eD_\rho} = - \sum_j P_j \left[1 + \frac{V_j - V_i}{2V_T}\right] Q_{ij} - \left(P_0 + \frac{P_i}{2}\right) \sum_j \frac{V_j}{V_T} R_{ij}. \quad (47)$$

It is to be understood that the summations are over noninsulated surfaces so the subscripts OTH to the summation symbols were left out.

To express the majority current (electrons in this case) in terms of the boundary values, it is helpful to define the following additional geometric constants.

For $i, j, k \in OTH$, define

$$S_{ijk} = \int_R \phi_i \vec{\nabla} \psi_j \cdot \vec{\nabla} \phi_k d^3X \quad (48)$$

$$T_{ijk} = \int_R \phi_i \vec{\nabla} \phi_j \cdot \vec{\nabla} \phi_k d^3X . \quad (49)$$

A relationship between T_{ijk} and R_{ij} can be obtained as follows. Green's theorem can be used to obtain

$$\begin{aligned} \int_R [\phi_k \nabla^2 (\phi_i \phi_j) - (\phi_i \phi_j) \nabla^2 \phi_k] d^3X \\ = \oint [\phi_k \vec{\nabla} (\phi_i \phi_j) - (\phi_i \phi_j) \vec{\nabla} \phi_k] \cdot d\vec{s} \end{aligned}$$

or

$$\begin{aligned} 2 \int_R \phi_k \vec{\nabla} \phi_i \cdot \vec{\nabla} \phi_j d^3X &= \oint [\phi_k \phi_i \vec{\nabla} \phi_j + \phi_k \phi_j \vec{\nabla} \phi_i - \phi_i \phi_j \vec{\nabla} \phi_k] \cdot d\vec{s} \\ &= \delta_{ik} \int_{A_k} \vec{\nabla} \phi_j \cdot d\vec{s} + \delta_{jk} \int_{A_k} \vec{\nabla} \phi_i \cdot d\vec{s} \\ &\quad - \delta_{ij} \int_{A_i} \vec{\nabla} \phi_k \cdot d\vec{s} . \end{aligned}$$

From (41) and (49) we have

$$2T_{kij} = \delta_{ik} R_{jk} + \delta_{jk} R_{ik} - \delta_{ij} R_{ik}$$

or

$$T_{ijk} = \frac{1}{2} [\delta_{ij} R_{jk} + \delta_{ik} R_{ij} - \delta_{jk} R_{ik}] . \quad (50)$$

One constraint between the constants S_{ijk} can be obtained by using the divergence theorem with

$$\oint \phi_i \cdot \psi_j \vec{\nabla} \phi_k \cdot d\vec{s} = \delta_{ij} \int_{A_i} \vec{\nabla} \phi_k \cdot d\vec{s} = \delta_{ij} R_{ik}$$

to get

$$\int_R \psi_j \vec{\nabla} \phi_i \cdot \vec{\nabla} \phi_k d^3X + \int_R \phi_i \vec{\nabla} \psi_j \cdot \vec{\nabla} \phi_k d^3X = \delta_{ij} R_{ik} .$$

Interchanging i and k and subtracting equations and then using (48) yield

$$S_{kji} = S_{ijk} + [\delta_{jk} - \delta_{ij}] R_{ik} . \quad (51)$$

Another constraint comes from (22) and (48) which immediately give

$$\sum_k S_{ijk} = 0 . \quad (52)$$

Equations (51) and (52) are the only independent constraints that this author has been able to find. If the reader is aware of any other constraints that are independent of (51) and (52), this author would appreciate being informed.

The electron current through the surface A_i is given by

$$I_{n_i} = \int_{A_i} \vec{j}_n \cdot d\vec{s}$$

or

$$\begin{aligned} \frac{I_{n_i}}{eD_n} &= \int_{A_i} \vec{\nabla} N \cdot d\vec{s} + \frac{1}{V_T} \int_{A_i} (N + n_o) \vec{E} \cdot d\vec{s} = \int_{A_i} \vec{\nabla} N \cdot d\vec{s} \\ &+ \frac{(N_i + n_o)}{V_T} \int_{A_i} \vec{E} \cdot d\vec{s} . \end{aligned}$$

Using (23) and (41) gives

$$\frac{I_{n_i}}{eD_n} = \int_{A_i} \vec{\nabla} N \cdot d\vec{s} - (N_i + n_o) \sum_j \frac{V_j}{V_T} R_{ij} . \quad (53)$$

Using (25b), (37), (40) and (41) gives the first order equality

$$\begin{aligned} \int_{A_i} \vec{\nabla} N \cdot d\vec{s} &= \int_{A_i} \vec{\nabla} \eta^{(o)} \cdot d\vec{s} + \sum_j V_j \int_{A_i} \vec{\nabla} \eta_j \cdot d\vec{s} \\ &= \sum_j \left[\frac{L_p^2}{L_n^2} P_j Q_{ij} + (N_j - \frac{L_p^2}{L_n^2} P_j) R_{ij} \right] + \sum_j V_j \int_{A_i} \vec{\nabla} \eta_j \cdot d\vec{s} . \end{aligned} \quad (54)$$

Note

$$\int_{A_i} \vec{\nabla} \eta_j \cdot d\vec{s} = \oint \phi_i \vec{\nabla} \eta_j \cdot d\vec{s} - \int_{INS} \phi_i \vec{\nabla} \eta_j \cdot d\vec{s}$$

where the subscript INS means to integrate on the insulated surfaces. But (32b) and (32d) show that on the insulated surfaces,

$$\vec{\nabla} \eta_j \cdot \hat{K} = \frac{L_p^2}{L_n^2} \vec{\nabla} \rho_j \cdot \hat{K}$$

so that

$$\int_{A_i} \vec{\nabla} \eta_j \cdot d\vec{s} = \oint \phi_i \vec{\nabla} \eta_j \cdot d\vec{s} - \frac{L_p^2}{L_n^2} \int_{INS} \phi_i \vec{\nabla} \rho_j \cdot d\vec{s} .$$

Since $\eta_j \vec{\nabla} \phi_i$ is zero on all surfaces, the equation can be written as

$$\int_{A_i} \vec{\nabla} \eta_j \cdot d\vec{s} = \oint [\phi_i \vec{\nabla} \eta_j - \eta_j \vec{\nabla} \phi_i] \cdot d\vec{s} - \frac{L_p^2}{L_n^2} \int_{INS} \phi_i \vec{\nabla} \rho_j \cdot d\vec{s}.$$

Similar steps produce

$$\int_{A_i} \vec{\nabla} \rho_j \cdot d\vec{s} = \oint [\phi_i \vec{\nabla} \rho_j - \rho_j \vec{\nabla} \phi_i] \cdot d\vec{s} - \int_{INS} \phi_i \vec{\nabla} \rho_j \cdot d\vec{s}.$$

Combining the above equations yields

$$\begin{aligned} \int_{A_i} \vec{\nabla} \eta_j \cdot d\vec{s} - \frac{L_p^2}{L_n^2} \int_{A_i} \vec{\nabla} \rho_j \cdot d\vec{s} &= \oint [\phi_i \vec{\nabla} \eta_j - \eta_j \vec{\nabla} \phi_i] \cdot d\vec{s} \\ &- \frac{L_p^2}{L_n^2} \oint [\phi_i \vec{\nabla} \rho_j - \rho_j \vec{\nabla} \phi_i] \cdot d\vec{s}. \end{aligned}$$

Green's theorem together with (27) and (29) give

$$\begin{aligned} \int_{A_i} \vec{\nabla} \eta_j \cdot d\vec{s} - \frac{L_p^2}{L_n^2} \int_{A_i} \vec{\nabla} \rho_j \cdot d\vec{s} &= \int_R \phi_i [\nabla^2 \eta_j - \frac{L_p^2}{L_n^2} \nabla^2 \rho_j] d^3x \\ &= \frac{1}{V_T} \int_R [\phi_i \vec{\nabla} \eta^{(0)} \cdot \vec{\nabla} \phi_j + \frac{L_p^2}{L_n^2} \phi_i \vec{\nabla} \rho^{(0)} \cdot \vec{\nabla} \phi_j] d^3x. \end{aligned}$$

Using (37), (39), (48) and (49), the equation becomes

$$\begin{aligned} \int_{A_i} \vec{\nabla} \eta_j \cdot d\vec{s} - \frac{L_p^2}{L_n^2} \int_{A_i} \vec{\nabla} \rho_j \cdot d\vec{s} &= \frac{2}{V_T} \frac{L_p^2}{L_n^2} \sum_k P_k S_{ikj} \\ &+ \frac{1}{V_T} \sum_k (N_k - \frac{L_p^2}{L_n^2} P_k) T_{ijk}. \end{aligned}$$

But, from (38),

$$\begin{aligned} \int_{A_i} \vec{\nabla} \rho_j \cdot d\vec{s} &= \frac{1}{2V_T} \int_{A_i} [P_j \vec{\nabla} \psi_j - \vec{\nabla}(\phi_j \sum_k P_k \psi_k)] \cdot d\vec{s} \\ &= \frac{1}{2V_T} \{P_j Q_{ij} - P_i R_{ij} - \delta_{ij} \sum_k P_k Q_{ik}\} \end{aligned}$$

so the equation becomes

$$\begin{aligned} \int_{A_i} \vec{\nabla} \eta_j \cdot d\vec{s} &= \frac{1}{V_T} \frac{L_P^2}{L_n^2} \left\{ \frac{1}{2} P_j Q_{ij} - \frac{1}{2} P_i R_{ij} - \frac{1}{2} \delta_{ij} \sum_k P_k Q_{ik} \right. \\ &\quad \left. + \frac{L_n^2}{L_P^2} \sum_k N_k T_{ijk} - \sum_k P_k T_{ijk} + 2 \sum_k P_k S_{ikj} \right\}. \end{aligned}$$

Combining this with (53) and (54) and using (50) finally produce

$$\begin{aligned} \frac{I_{n_i}}{eD_n} &= -\left(\frac{N_i}{2} + n_o + \frac{D_P}{D_n} P_i\right) \sum_j \frac{V_j}{V_T} R_{ij} + \sum_j (N_j - \frac{D_P}{D_n} P_j) \left[1 + \frac{V_i - V_j}{2V_T}\right] R_{ij} \\ &\quad + \frac{D_P}{D_n} \sum_j P_j \left[1 + \frac{V_j - V_i}{2V_T}\right] Q_{ij} + 2 \frac{D_P}{D_n} \sum_{j,k} \frac{V_j}{V_T} P_k S_{ikj} \end{aligned} \quad (55)$$

Equation (55) is a cumbersome way to express the electron current. A simpler expression can be obtained by making additional approximations which are based on the excess electron density being very small compared to the equilibrium density. In order to make use of this, it is necessary to be able to identify the quantities that are very small. To first order, the drift current is given by

$$\begin{aligned} (\vec{J}_n)_{\text{drift}} &= \frac{eD_n}{V_T} (N + n_o) \vec{E} = \frac{eL_n^2}{\tau V_T} (N + n_o) \vec{E} \\ (I_{n_i})_{\text{drift}} &= \frac{eL_n^2}{\tau V_T} \int_{A_i} (N + n_o) \vec{E} \cdot d\vec{s} = \frac{eL_n^2(N_i + n_o)}{\tau V_T} \int_{A_i} \vec{E} \cdot d\vec{s}. \end{aligned}$$

It is obvious that

$$\frac{eL_n^2 N_i}{\tau V_T} \int_{A_i} \vec{E} \cdot d\vec{s} = \frac{-eL_n^2}{\tau V_T} N_i \sum_j V_j \int_{A_i} \vec{\nabla} \phi_j \cdot d\vec{s}$$

is a very small contribution to the drift current. It is reasonable to expect that this quantity is negligible compared to the total electron current I_{n_i} . It is also reasonable to expect that if we sum the terms over i , the result

$$\frac{-eL_n^2}{\tau V_T} \sum_j V_j \sum_k N_k \int_{A_k} \vec{\nabla} \phi_j \cdot d\vec{s} \quad (56)$$

is small compared to I_{n_i} . Note that

$$\sum_k N_k \int_{A_k} \vec{\nabla} \phi_j \cdot d\vec{s} = \oint \eta^{(0)} \vec{\nabla} \phi_j \cdot d\vec{s} = \int_R \vec{\nabla} \eta^{(0)} \cdot \vec{\nabla} \phi_j d^3X$$

so the expression (56) becomes

$$\frac{-eL_n^2}{\tau V_T} \sum_j V_j \int_R \vec{\nabla} \eta^{(0)} \cdot \vec{\nabla} \phi_j d^3X$$

which can also be identified as a small quantity. This should be small compared to I_{n_i} regardless of how the V_j 's are chosen so the terms

$$\frac{eL_n^2}{\tau V_T} V_j \int_R \vec{\nabla} \eta^{(0)} \cdot \vec{\nabla} \phi_j d^3X \quad (57)$$

should individually be small. Note that since ϕ_k satisfies Laplace's equation in R , its maximum and minimum values are on the boundary surface of R . The boundary values are zero and one so ϕ_k should be between zero and one everywhere in R (this isn't strictly correct because the boundary values were not specified on the insulated surfaces, but it is still reasonable to expect ϕ_k to be on the order of one or less everywhere). But if the expression (57)

is small and if ϕ_k is on the order of one or less, it is reasonable to expect that the quantity

$$\frac{eL_n^2}{\tau V_T} V_j \int_R \phi_k \vec{\nabla} \eta^{(0)} \cdot \vec{\nabla} \phi_j d^3X$$

is also small compared to I_{n_i} . Using (39), (48) and (49), the above

expression can be written as

$$\frac{eL_n^2 V_j}{\tau V_T} \left\{ \frac{L_p^2}{L_n^2} \sum_{\ell} P_{\ell} S_{k\ell j} + \sum_{\ell} (N_{\ell} - \frac{L_p^2}{L_n^2} P_{\ell}) T_{k\ell j} \right\}.$$

But if the above expression is small for independently chosen boundary values, the individual terms must be small, i.e.,

$$\frac{eL_p^2 V_j}{\tau V_T} P_{\ell} S_{k\ell j} \quad \text{and} \quad (58)$$

$$\frac{eL_n^2 V_j}{\tau V_T} (N_{\ell} - \frac{L_p^2}{L_n^2} P_{\ell}) T_{k\ell j} \quad (59)$$

can each be identified as small quantities. Expression (56) and equation (41) produce other small quantities which are

$$\frac{eL_n^2}{\tau V_T} N_i V_j R_{ij}. \quad (60)$$

Combining (53), (54) and (55) and adding or subtracting terms that are of the form (58), (59) or (60) result in the approximation

$$\frac{1}{eD_n} I'_{n_i} = \frac{L_p^2}{L_n^2} \sum_j P_j \left[1 + \frac{V_j - V_i}{2V_T} \right] Q_{ij} + \sum_j R_{ij} \left[N_j + \frac{L_p^2}{L_n^2} P_j \left(\frac{V_j}{2V_T} - 1 \right) - \frac{V_j}{V_T} n_o \right]. \quad (61)$$

The only motivation for including a small quantity of the form (60) in the above equation is to make the total current (electron plus hole current), as predicted by these equations, add up to exactly zero when summed over all noninsulated surfaces.

The important equations for both n and p type materials are listed in Table 5-1.

Table 5-1. Summary

A_i denotes the i^{th} section of the boundary surface of a quasineutral homogeneous region R . The index sets INS , OTH are defined by:

If $i \in INS$ then A_i is an insulated surface
 If $i \in OTH$ then A_i is a noninsulated surface

Other symbols are defined as follows:

N = Excess electron density

P = Excess hole density

n_0 = Equilibrium electron density

p_0 = Equilibrium hole density

τ = Recombination Time

L_n, L_p = Diffusion lengths for electrons and holes respectively

D_n, D_p = Diffusion constants for electrons and holes respectively

$V_T = KT/e$ = Thermal voltage

e = Elementary charge

\hat{K} = Normal unit vector directed outwards from the region R .

I_{n_i}, I_{p_i}, I_i = Electron, hole, and total currents respectively through the i^{th} noninsulated surface. These currents are directed outwards from the region R .

\vec{X} = Position vector representing the point of evaluation of a function.

δ_{ij} = Kronecker delta

Boundary conditions that are assumed for the carrier densities and applied potential are:

If $i \in INS$ then the surface A_i has a recombination velocity S_i .

If $i \in OTH$ then the surface A_i has constant applied potential and constant excess carrier densities denoted by V_i, N_i, P_i .

All summation symbols denote sums over the index set OTH . All surface integrals refer to a unit normal vector (K) directed outwards from the region R .

Table 5-1. Summary (contd)

If R is an n Type Material

For each $j \in \text{OTH}$, define the functions ϕ_j, ψ_j as solutions to the following boundary value problems:

$$\nabla^2 \phi_j = 0 \text{ and } \nabla^2 \psi_j = \frac{\psi_j}{L_p^2} \text{ in } R.$$

If $i \in \text{INS}$ and \vec{X} is on A_i then

$$\vec{\nabla} \phi_j \cdot \hat{K} = 0 \text{ and } \vec{\nabla} \psi_j \cdot \hat{K} = \frac{-S_i \tau}{L_p^2} \psi_j.$$

If $i \in \text{OTH}$ and \vec{X} is on A_i then

$$\phi_j = \delta_{ij} \text{ and } \psi_j = \delta_{ij}.$$

To first order in the applied voltages,

$$P = \rho^{(0)} + \sum_j V_j \rho_j$$

where

$$\rho^{(0)} = \sum_j P_j \psi_j \text{ and } \rho_j = \frac{1}{2V_T} [P_j \psi_j - \phi_j \sum_k P_k \psi_k].$$

Define the constants

$$Q_{ij} = \int_{A_i} \vec{\nabla} \psi_j \cdot d\vec{s} \quad R_{ij} = \int_{A_i} \vec{\nabla} \phi_j \cdot d\vec{s}$$

(Note that $Q_{ij} = Q_{ji}$, $R_{ij} = R_{ji}$ and $\sum_j R_{ij} = 0$).

Table 5-1. Summary (contd)

To first order we have

$$\frac{I_{P_i}}{eD_p} = -\sum_j P_j \left[1 + \frac{V_j - V_i}{2V_T}\right] Q_{ij} - \left(P_o + \frac{P_i}{2}\right) \sum_j \frac{V_j}{V_T} R_{ij}.$$

Define the constants

$$S_{ijk} = \int_R \phi_i \vec{\nabla} \psi_j \cdot \vec{\nabla} \phi_k d^3X \quad T_{ijk} = \int_R \phi_i \vec{\nabla} \phi_j \cdot \vec{\nabla} \phi_k d^3X$$

(Note that $T_{ijk} = \frac{1}{2} [\delta_{ij} R_{jk} + \delta_{ik} R_{ij} - \delta_{jk} R_{ik}]$ and $S_{kji} = S_{ijk} + [\delta_{jk} - \delta_{ij}] R_{ik}$ and $\sum_k S_{ijk} = 0$).

To first order we have

$$\begin{aligned} \frac{I_{n_i}}{eD_n} = & -\left(\frac{N_i}{2} + n_o + \frac{D_p}{D_n} P_i\right) \sum_j \frac{V_j}{V_T} R_{ij} + \sum_j (N_j - \frac{D_p}{D_n} P_j) \left[1 + \frac{V_i - V_j}{2V_T}\right] R_{ij} \\ & + \frac{D_p}{D_n} \sum_j P_j \left[1 + \frac{V_j - V_i}{2V_T}\right] Q_{ij} + 2 \frac{D_p}{D_n} \sum_{j,k} \frac{V_j}{V_T} P_k S_{ikj} \end{aligned}$$

and

$$\begin{aligned} \frac{I_i}{e} = & -\left(\frac{1}{2} D_n N_i + D_n n_o + \frac{3}{2} D_p P_i + D_p P_o\right) \sum_j \frac{V_j}{V_T} R_{ij} \\ & + \sum_j (D_n N_j - D_p P_j) \left[1 + \frac{V_i - V_j}{2V_T}\right] R_{ij} + 2 \sum_{j,k} \frac{V_j}{V_T} D_p P_k S_{ikj}. \end{aligned}$$

Table 5-1. Summary (contd)

If $N_i, P_i, \ll n_o$, an approximation for the current is

$$\frac{I_{ni}}{eD_n} = \frac{L_p^2}{L_n^2} \sum_j P_j \left[1 + \frac{V_j - V_i}{2V_T} \right] Q_{ij} + \sum_j R_{ij} \left[N_j + \frac{L_p^2}{L_n^2} P_j \left(\frac{V_i}{2V_T} - 1 \right) - \frac{V_i}{V_T} n_o \right]$$

If R is a p Type Material

For each $j \in OTH$, define the functions ϕ_j, ψ_j as solutions to the following boundary value problems:

$$\nabla^2 \phi_j = 0 \text{ and } \nabla^2 \psi_j = \frac{\psi_i}{L_n^2} \text{ in } R.$$

If $i \in INS$ and \vec{X} is on A_i then

$$\vec{\nabla} \phi_j \cdot \hat{K} = 0 \text{ and } \vec{\nabla} \psi_j \cdot \hat{K} = \frac{-S_i \tau}{L_n^2} \psi_j.$$

If $i \in OTH$ and \vec{X} is on A_i then

$$\phi_j = \delta_{ij} \text{ and } \psi_j = \delta_{ij}.$$

To first order in the applied voltages,

$$N = n^{(o)} + \sum_j V_j n_j$$

Table 5-1. Summary (contd)

where

$$\eta^{(0)} = \sum_j N_j \psi_j \text{ and } \eta_j = \frac{1}{2V_T} [\phi_j \sum_k N_k \psi_k - N_j \psi_j] .$$

Define the constants

$$Q_{ij} = \int_{A_i} \vec{\nabla} \psi_j \cdot d\vec{s} \quad R_{ij} = \int_{A_i} \vec{\nabla} \phi_j \cdot d\vec{s}$$

(Note that $Q_{ij} = Q_{ji}$, $R_{ij} = R_{ji}$ and $\sum_j R_{ij} = 0$) .

To first order we have

$$\frac{I_{n_i}}{eD_n} = \sum_j N_j \left[1 + \frac{V_i - V_j}{2V_T} \right] Q_{ij} - \left(n_o + \frac{N_i}{2} \right) \sum_j \frac{V_j}{V_T} R_{ij} .$$

Define the constants

$$S_{ijk} = \int_R \phi_i \vec{\nabla} \psi_j \cdot \vec{\nabla} \phi_k d^3x \quad T_{ijk} = \int_R \phi_i \vec{\nabla} \phi_j \cdot \vec{\nabla} \phi_k d^3x$$

(Note that $T_{ijk} = \frac{1}{2} [\delta_{ij} R_{jk} + \delta_{ik} R_{ij} - \delta_{jk} R_{ik}]$ and $S_{kji} = S_{ijk} + [\delta_{jk} - \delta_{ij}] R_{ik}$ and $\sum_k S_{ijk} = 0$).

To first order we have

$$\begin{aligned} \frac{I_{p_i}}{eD_p} = & -\left(\frac{P_i}{2} + P_o + \frac{D_n}{D_p} N_i\right) \sum_j \frac{V_j}{V_T} R_{ij} - \sum_j \left(P_j - \frac{D_n}{D_p} N_j\right) \left[1 + \frac{V_j - V_i}{2V_T}\right] R_{ij} \\ & - \frac{D_n}{D_p} \sum_j N_j \left[1 + \frac{V_i - V_j}{2V_T}\right] Q_{ij} + 2 \frac{D_n}{D_p} \sum_{j,k} \frac{V_j}{V_T} N_k S_{ikj} \end{aligned}$$

Table 5-1. Summary (contd)

and

$$\begin{aligned} \frac{I_i}{e} = & - \left(\frac{1}{2} D_p P_i + D_p P_o + \frac{3}{2} D_n N_i + D_n n_o \right) \sum_j \frac{V_j}{V_T} R_{ij} \\ & - \sum_j (D_p P_j - D_n N_j) \left[1 + \frac{V_j - V_i}{2V_T} \right] R_{ij} + 2 \sum_{j,k} \frac{V_j}{V_T} D_n N_k S_{ikj}. \end{aligned}$$

If $N_i, P_i \ll p_o$, an approximation for the current is

$$\begin{aligned} \frac{I_{P_i}}{eD_p} = & - \frac{L_n^2}{L_p^2} \sum_j N_j \left[1 + \frac{V_i - V_j}{2V_T} \right] Q_{ij} \\ & + \sum_j R_{ij} \left[-P_j + \frac{L_n^2}{L_p^2} N_j \left(\frac{V_i}{2V_T} + 1 \right) - \frac{V_j}{V_T} P_o \right]. \end{aligned}$$

SECTION 6

RECOMMENDATIONS FOR FUTURE WORK

The analysis presented was intended to be used as an analytical tool for the study of latches but the study was not completed. It is expected that this analytical tool will have some useful applications in its present stage of development but some important work remains to be done before this tool can be applied to the study of latches.

Assuming that the first order perturbation method is valid, the equations listed in the summary table have general applicability to homogeneous quasi-neutral regions providing the boundary surface is partitioned in such a way that p , n and ϕ are approximately constant on each noninsulated section. What is probably the most obvious weakness in the application of these equations is the assumption that an entire depletion region boundary surface is a constant p , n , ϕ surface. The accuracy could be improved by partitioning the depletion region boundary surface into a few sections with p , n and ϕ treated as constant on each section. However, such a partitioning would require that we properly account for the tangential currents that flow inside the depletion region between adjacent sections. It was not shown how to do this and this remains to be done.

A time dependent analysis using low frequency approximations (where steady state equations are applied to the interior of the region but DC capacitances are associated with various boundary surfaces so that terms containing time derivatives of voltages are added to various currents) should be relatively straightforward but it was not done here. It is hoped that this will be done in future work.

Examples which illustrate the use of the equations have not been included in this publication. It is expected that this will be done in future work.

Last, but obviously not least, a comparison is needed between measurement and theory. This is also expected to be done in future work.

Some other assumptions that were made are not essential to the theory and are therefore not considered to be a weakness of the theory. These include neglecting recombination in the space charge regions, neglecting the possibility of avalanching in the depletion regions and assuming that junctions are abrupt. These characteristics have no effect on the equations listed in the summary table, which is the principal product of this publication. These characteristics affect only the relationships between the boundary values. Relationships that include such effects as avalanching, recombination or graded junctions could be used with the equations in the summary table without modification of those equations.

SECTION 7

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APPENDIX A
HIGH-LOW JUNCTIONS AND DEPLETION REGIONS

High-low junctions and depletion regions are two examples of the general condition of a doping density that is variable in a narrow region of space which results in a reasonably well defined space charge region. The analysis to follow applies to this general category providing that certain approximations are valid.

We assume that a one dimensional analysis can be used and take the x axis to be perpendicular to the junction. Generalized transport equations have been provided by Overstraeten, DeMan and Mertens [5]. When applied to the interior of the space charge region where the doping density is nonuniform, the equations contain an additional drift term and are written as

$$J_{p,x} = eD_p \left[-\frac{dp}{dx} - \frac{p}{V_T} \frac{d}{dx} (\phi_t - \text{DEV}) \right] \quad (\text{A-1})$$

$$J_{n,x} = eD_n \left[\frac{dn}{dx} - \frac{n}{V_T} \frac{d}{dx} (\phi_t + \text{DEC}) \right] \quad (\text{A-2})$$

where ϕ_t is the total (as opposed to applied) electrostatic potential and DEV and DEC are functions of the doping concentrations. The DEV and DEC used here differ from those used by Overstraeten et al by a factor of the electronic charge. It is assumed that the net (diffusion plus drift) currents for each carrier are small compared to the individual contributions [3]. This requires negligible recombination in the space charge region. The result is

$$\frac{dp}{dx} = -\frac{p}{V_T} \frac{d}{dx} (\phi_t - \text{DEV})$$

$$\frac{dn}{dx} = \frac{n}{V_T} \frac{d}{dx} (\phi_t + \text{DEC}) .$$

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If the boundaries of the space charge region are at x_1 and x_2 , the equations can be integrated to give

$$\frac{p(x_2)}{p(x_1)} = \exp [(\phi_t(x_1) - \phi_t(x_2) + \text{DEV}(x_2) - \text{DEV}(x_1))/V_T] \quad (\text{A-3})$$

$$\frac{n(x_2)}{n(x_1)} = \exp [(\phi_t(x_2) - \phi_t(x_1) + \text{DEC}(x_2) - \text{DEC}(x_1))/V_T] . \quad (\text{A-4})$$

Let p_o , n_o , ϕ_o denote the equilibrium densities and potential. Applying (A-3) and (A-4) to the equilibrium conditions gives

$$\frac{p_o(x_2)}{p_o(x_1)} = \exp [(\phi_o(x_1) - \phi_o(x_2) + \text{DEV}(x_2) - \text{DEV}(x_1))/V_T]$$

$$\frac{n_o(x_2)}{n_o(x_1)} = \exp [(\phi_o(x_2) - \phi_o(x_1) + \text{DEC}(x_2) - \text{DEC}(x_1))/V_T]$$

so that (A-3) and (A-4) become

$$\frac{p(x_2)}{p(x_1)} = \frac{p_o(x_2)}{p_o(x_1)} \exp [(\phi(x_1) - \phi(x_2))/V_T] \quad (\text{A-5})$$

$$\frac{n(x_2)}{n(x_1)} = \frac{n_o(x_2)}{n_o(x_1)} \exp [(\phi(x_2) - \phi(x_1))/V_T] \quad (\text{A-6})$$

where

$$\phi(x) = \phi_t(x) - \phi_o(x) \quad (\text{A-7})$$

is the applied potential. To shorten the notation, define β by

$$\beta = \exp [(\phi(x_1) - \phi(x_2))/V_T] \quad (\text{A-8})$$

so that (A-5) and (A-6) become

$$\frac{p(x_2)}{p(x_1)} = \frac{p_o(x_2)}{p_o(x_1)} \beta \quad (\text{A-9})$$

$$\frac{n(x_2)}{n(x_1)} = \frac{n_o(x_2)}{n_o(x_1)} \beta^{-1} . \quad (\text{A-10})$$

Let $N_D(x)$ and $N_A(x)$ be the density of donor and acceptor ions. Assuming space charge neutrality at x_1 and x_2 we get

$$p(x_1) - n(x_1) = N_A(x_1) - N_D(x_1)$$

$$p(x_2) - n(x_2) = N_A(x_2) - N_D(x_2) .$$

Combining these equations with (A-9) and (A-10) gives

$$p(x_1) = \frac{\frac{n_o(x_2)}{n_o(x_1)} \beta^{-1} [N_D(x_1) - N_A(x_1)] - [N_D(x_2) - N_A(x_2)]}{\frac{p_o(x_2)}{p_o(x_1)} \beta - \frac{n_o(x_2)}{n_o(x_1)} \beta^{-1}} \quad (\text{A-11})$$

$$p(x_2) = \frac{\frac{n_o(x_1)}{n_o(x_2)} \beta [N_D(x_2) - N_A(x_2)] - [N_D(x_1) - N_A(x_1)]}{\frac{p_o(x_1)}{p_o(x_2)} \beta^{-1} - \frac{n_o(x_1)}{n_o(x_2)} \beta} \quad (\text{A-12})$$

$$n(x_1) = \frac{\frac{p_o(x_2)}{p_o(x_1)} \beta [N_A(x_1) - N_D(x_1)] - [N_A(x_2) - N_D(x_2)]}{\frac{n_o(x_2)}{n_o(x_1)} \beta^{-1} - \frac{p_o(x_2)}{p_o(x_1)} \beta} \quad (A-13)$$

$$n(x_2) = \frac{\frac{p_o(x_1)}{p_o(x_2)} \beta^{-1} [N_A(x_2) - N_D(x_2)] - [N_A(x_1) - N_D(x_1)]}{\frac{n_o(x_1)}{n_o(x_2)} \beta - \frac{p_o(x_1)}{p_o(x_2)} \beta^{-1}} . \quad (A-14)$$

Since recombination in the space charge region has been neglected, we also have

$$J_{p,x}(x_1) = J_{p,x}(x_2) \quad (A-15)$$

$$J_{n,x}(x_1) = J_{n,x}(x_2) \quad (A-16)$$

If the equilibrium carrier densities can be regarded as known, equations (A-11) through (A-16) constitute a complete system of equations describing the boundary conditions. But to use these equations in a practical problem, a method of estimating the equilibrium densities from existing empirical data is needed. Slotboom and De Graaff [4] have related the equilibrium densities to the intrinsic densities (the equilibrium densities of the material without doping) through the equation

$$p_o n_o = n_i^2 \exp [\Delta V_{go}/V_T]$$

where n_i is the intrinsic carrier density and ΔV_{go} is an empirically determined function of the doping level. Empirical values are given by Slotboom and De Graaff. Applying this equation to x_1 and x_2 yields

$$p_o(x_1) n_o(x_1) = n_i^2 \exp [\Delta V_{go}(x_1)/V_T] \quad (A-17)$$

$$p_o(x_2) n_o(x_2) = n_i^2 \exp [\Delta V_{go}(x_2)/V_T] . \quad (A-18)$$

If the junction is abrupt with the (known) doping concentration uniform on each side of the point of discontinuity, the doping concentrations at x_1 and x_2 are known without the need of solving for x_1 and x_2 (for a more general doping profile, x_1 and x_2 would have to be solved and their values would depend on the doping profile) and $\Delta V_{go}(x_1)$, $\Delta V_{go}(x_2)$ can be immediately evaluated from the data provided by Slotboom and De Graaff. By assuming that the majority carrier density is approximately equal to the doping density, (A-17) and (A-18) can be used to solve for the minority carrier densities at x_1 and x_2 . The equilibrium minority carrier densities can now be regarded as known. Approximating the majority densities with the doping densities may be adequate for the purpose of using (A-17) and (A-18) to solve for the minority densities, but it is not sufficiently accurate for other calculations for the following reason. If the equilibrium majority carrier density is set equal to the doping density, the error in the equilibrium majority carrier density will be equal to the equilibrium minority carrier density. If this error ultimately showed up in the nonequilibrium majority carrier density in the form of an additive constant, it would not affect the calculation of the diffusion current and it would negligibly affect the calculation of the drift current, so the error would be minor. But the error will show up in the nonequilibrium density in a way that is more serious than the error resulting from an additive constant. This is because an error in the equilibrium density will affect the boundary values of the nonequilibrium excess carrier density as calculated by (A-11) through (A-14). This error is small compared to the majority carrier density, but it may not be small compared to the excess majority carrier density. In fact, it is not difficult to show that the error in the calculated boundary value for the nonequilibrium excess majority carrier density (as calculated by (A-11) through (A-14)) exceeds the error in the equilibrium density. When the diffusion equations are used to calculate the excess density as a function of location, errors in the boundary values affect the solution in a nontrivial way and the calculated currents will have serious errors. Therefore, to calculate the equilibrium majority carrier density, we must be more accurate than simply setting it equal to the doping density. A more accurate treatment uses space charge neutrality to get

$$p_o(x_1) - n_o(x_1) = N_A(x_1) - N_D(x_1) \quad (A-19)$$

$$p_o(x_2) - n_o(x_2) = N_A(x_2) - N_D(x_2) . \quad (A-20)$$

Note that this space charge neutrality condition is needed in order for (A-11) through (A-14) to predict the correct equilibrium ($\beta=1$) values for the carrier densities.

Example 1: High-Low Junction

Consider an $n - n^+$ junction and let x_2 be on the heavily doped side. To use more suggestive notation, let

$$x_h = x_2, x_l = x_1, N_{Dh} = N_D(x_2), N_{Dl} = N_D(x_1)$$

$$p_{oh} = p_o(x_2), p_{ol} = p_o(x_1), n_{oh} = n_o(x_2), n_{ol} = n_o(x_1)$$

$$p_h = p(x_2), p_l = p(x_1), n_h = n(x_2), n_l = n(x_1)$$

$$\phi_h = \phi(x_2), \phi_l = \phi(x_1) .$$

Equations (A-11) through (A-14) become

$$p_l = \frac{\frac{n_{oh}}{n_{ol}} \beta^{-1} N_{Dl} - N_{Dh}}{\frac{p_{oh}}{p_{ol}} \beta - \frac{n_{oh}}{n_{ol}} \beta^{-1}} \quad (A-21)$$

$$p_h = \frac{\frac{n_{ol}}{n_{oh}} \beta N_{Dh} - N_{Dl}}{\frac{p_{ol}}{p_{oh}} \beta^{-1} - \frac{n_{ol}}{n_{oh}} \beta} \quad (A-22)$$

$$n_l = \frac{-\frac{p_{oh}}{p_{ol}} \beta N_{Dl} + N_{Dh}}{\frac{n_{oh}}{n_{ol}} \beta^{-1} - \frac{p_{oh}}{p_{ol}} \beta} \quad (A-23)$$

$$n_h = \frac{-\frac{p_{ol}}{p_{oh}} \beta^{-1} N_{Dh} + N_{Dl}}{\frac{n_{ol}}{n_{oh}} \beta - \frac{p_{ol}}{p_{oh}} \beta^{-1}} \quad (A-24)$$

where

$$\beta = \exp [(\phi_l - \phi_h)/V_T] . \quad (A-25)$$

Assume that ΔV_{go} is essentially zero on the low doped side and represent its value as simply ΔV_{go} on the high doped side. Equations (A-17) and (A-18) become

$$p_{ol} N_{Dl} = n_i^2$$

$$p_{oh} N_{Dh} = n_i^2 \exp [\Delta V_{go}/V_T] .$$

Combining these equations with (A-19) and (A-20) yields

$$n_{ol} = N_{Dl} + \frac{n_i^2}{N_{Dl}} \quad (A-26)$$

$$n_{oh} = N_{Dh} + \frac{n_i^2}{N_{Dh}} \exp [\Delta V_{go}/V_T] . \quad (A-27)$$

Using the same analysis for a p - P⁺ junction yields

$$p_l = \frac{-\frac{n_{oh}}{n_{ol}} \beta^{-1} N_{Al} + N_{Ah}}{\frac{p_{oh}}{p_{ol}} \beta - \frac{n_{oh}}{n_{ol}} \beta^{-1}}$$

$$p_h = \frac{-\frac{n_{ol}}{n_{oh}} \beta N_{Ah} + N_{Al}}{\frac{p_{ol}}{p_{oh}} \beta^{-1} - \frac{n_{ol}}{n_{oh}} \beta}$$

$$n_l = \frac{\frac{p_{oh}}{p_{ol}} \beta N_{Al} - N_{Ah}}{\frac{n_{oh}}{n_{ol}} \beta^{-1} - \frac{p_{oh}}{p_{ol}} \beta}$$

$$n_h = \frac{\frac{p_{ol}}{p_{oh}} \beta^{-1} N_{Ah} - N_{Al}}{\frac{n_{ol}}{n_{oh}} \beta - \frac{p_{ol}}{p_{oh}} \beta^{-1}}$$

$$n_{ol} = \frac{n_i^2}{N_{Al}}$$

$$n_{oh} = \frac{n_i^2}{N_{Ah}} \exp [\Delta V_{go}/V_T]$$

$$p_{ol} = N_{Al} + \frac{n_i^2}{N_{Al}}$$

$$p_{oh} = N_{Ah} + \frac{n_i^2}{N_{Ah}} \exp [\Delta V_{go}/V_T] .$$

Example 2: PN Junction

Let x_1 be on the p side and x_2 be on the n side. To use more suggestive notation, let

$$x_p = x_1, x_n = x_2, N_{Ap} = N_A(x_1), N_{Dn} = N_D(x_2)$$

$$p_{op} = p_o(x_1), p_{on} = p_o(x_2), n_{op} = n_o(x_1), n_{on} = n_o(x_2)$$

$$p_p = p(x_1), p_n = p(x_2), n_p = n(x_1), n_n = n(x_2)$$

$$\phi_p = \phi(x_1), \phi_n = \phi(x_2), \Delta V_p = \Delta V_{go}(x_1), \Delta V_n = \Delta V_{go}(x_2) .$$

Equations (A-11) through (A-14) become

$$p_p = \frac{\frac{n_{on}}{n_{op}} \beta^{-1} N_{Ap} + N_{Dn}}{\frac{n_{on}}{n_{op}} \beta^{-1} - \frac{p_{on}}{p_{op}} \beta}$$

$$p_n = \frac{\frac{n_{op}}{n_{on}} \beta N_{Dn} + N_{Ap}}{\frac{p_{op}}{p_{on}} \beta^{-1} - \frac{n_{op}}{n_{on}} \beta}$$

$$n_p = \frac{\frac{p_{on}}{p_{op}} \beta N_{Ap} + N_{Dn}}{\frac{n_{on}}{n_{op}} \beta^{-1} - \frac{p_{on}}{p_{op}} \beta}$$

$$n_n = \frac{\frac{p_{op}}{p_{on}} \beta^{-1} N_{Dn} + N_{Ap}}{\frac{p_{op}}{p_{on}} \beta^{-1} - \frac{n_{op}}{n_{on}} \beta}$$

with

$$\beta = \exp [(\phi_p - \phi_n)/V_T]$$

while (A-17) through (A-20) imply

$$p_{on} = \frac{n_i^2}{N_{Dn}} \exp [\Delta V_n / V_T]$$

$$n_{op} = \frac{n_i^2}{N_{Ap}} \exp [\Delta V_p / V_T]$$

$$p_{op} = N_{Ap} + \frac{n_i^2}{N_{Ap}} \exp [\Delta V_p / V_T]$$

$$n_{on} = N_{Dn} + \frac{n_i^2}{N_{Dn}} \exp [\Delta V_n / V_T] .$$

APPENDIX B
A UNIQUENESS THEOREM

Let a region R be bounded by a surface A. Consider the boundary value problem

$$(\nabla^2 - K)\psi(\vec{X}) = f(\vec{X}) \text{ for } \vec{X} \in R$$

$$a(\vec{X})\psi(\vec{X}) + b(\vec{X}) \vec{\nabla}\psi(\vec{X}) \cdot \hat{n} = g(\vec{X}) \text{ for } \vec{X} \in A$$

where f and g are specified functions and \hat{n} is the outer normal unit vector. K is a constant satisfying

$$K \geq 0 . \quad (B-1)$$

At any point \vec{X} on the surface A, the functions a and b are to satisfy one of the following conditions:

Either

$$a(\vec{X}) = 0 \text{ and } b(\vec{X}) \neq 0 \quad (B-2)$$

or

$$a(\vec{X}) \neq 0 \text{ and } b(\vec{X}) = 0 \quad (B-3)$$

or

$$a(\vec{X}) \neq 0 \text{ and } b(\vec{X}) \neq 0 \text{ and } a(\vec{X})/b(\vec{X}) > 0 . \quad (B-4)$$

It will be shown that the solution ψ is unique. Note that this is not the most general possible uniqueness theorem because of the conditions (B-2), (B-3), and (B-4) (there are also uniqueness theorems, for example, to the Helmholtz equation, which does not satisfy (B-1)) but it is general enough for our purposes since recombination velocities and diffusion lengths are positive quantities.

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Let ψ_1 and ψ_2 be two solutions to the boundary value problem and let

$$\psi' = \psi_2 - \psi_1 . \quad (\text{B-5})$$

Note

$$(\nabla^2 - K) \psi' = 0 \text{ for } \vec{X} \in R \quad (\text{B-6})$$

$$a(\vec{X}) \psi'(\vec{X}) + b(\vec{X}) \vec{\nabla} \psi'(\vec{X}) \cdot \hat{n} = 0 \text{ for } \vec{X} \in A . \quad (\text{B-7})$$

Green's first identity states that

$$\int_R \psi' \nabla^2 \psi' d^3X + \int_R \vec{\nabla} \psi' \cdot \vec{\nabla} \psi' d^3X = \int_A \psi' \vec{\nabla} \psi' \cdot d\vec{s} .$$

Using (B-6) to replace $\nabla^2 \psi'$ in the first integral gives

$$K \int_R (\psi')^2 d^3X + \int_R |\vec{\nabla} \psi'|^2 d^3X = \int_A \psi' \vec{\nabla} \psi' \cdot d\vec{s} . \quad (\text{B-8})$$

Let A_0 denote the section of the surface A such that either (B-2) or (B-3) is satisfied. Note that (B-7) combined with either of these conditions implies that

$$\psi' \vec{\nabla} \psi' \cdot \hat{n} = 0 \text{ for } \vec{X} \in A_0 .$$

Therefore,

$$\int_A \psi' \vec{\nabla} \psi' \cdot d\vec{s} = \int_{A'} \psi' \vec{\nabla} \psi' \cdot d\vec{s}$$

where A' is the section of A such that (B-4) is satisfied. Using (B-7) gives

$$\int_A \psi' \vec{\nabla} \psi' \cdot d\vec{s} = - \int_{A'} \frac{a(\vec{X})}{b(\vec{X})} (\psi')^2 ds$$

so that (B-8) becomes

$$K \int_R (\psi')^2 d^3X + \int_R |\vec{\nabla} \psi'|^2 d^3X + \int_A \frac{a}{b} (\psi')^2 ds = 0 .$$

From (B-1) and (B-4) it is seen that the left side of the above equation is a sum of nonnegative quantities. Therefore, the equation implies

$$\psi' = 0$$

or, from (B-5),

$$\psi_1 = \psi_2 .$$